

# Time-of-arrival distributions from position-momentum and energy-time joint measurements.

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The position-momentum quasi-distribution obtained from an Arthurs and Kelly joint measurement model is used to obtain indirectly an “operational” time-of-arrival (TOA) distribution following a quantization procedure proposed by Kocharński and Wódkiewicz [Phys. Rev. A **60**, 2689 (1999)]. This TOA distribution is not time covariant. The procedure is generalized by using other phase-space quasi-distributions, and sufficient conditions are provided for time covariance that limit the possible phase-space quasi-distributions essentially to the Wigner function, which, however, provides a non-positive TOA quasi-distribution. These problems are remedied with a different quantization procedure which, on the other hand, does not guarantee normalization. Finally an Arthurs and Kelly measurement model for TOA and energy (valid also for arbitrary conjugate variables when one of the variables is bounded from below) is worked out. The marginal TOA distribution so obtained, a distorted version of Kijowski’s distribution, is time covariant, positive, and normalized.

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## I. INTRODUCTION

In 1965 Arthurs and Kelly [1] proposed a model for the simultaneous measurement of the position and momentum of a quantum particle. This model has been a valuable tool in many fundamental and applied works, see e.g. [2–10] and references therein, and is present, albeit implicitly, in many others [11–14]. In the last few years, the advances in quantum optics have led to experimental techniques and theoretical approaches devoted to the simultaneous measurement of conjugate variables that have made possible a practical realization of the original gedanken experiment of Arthurs and Kelly [2–4,15,16]. The measurement process is modeled by a Hamiltonian that includes degrees of freedom of the particle and two pointers. It is constructed by adding two sudden interaction terms that, when acting separately, provide impulsive (or von Neumann) measurements [17] of the position and the momentum of the particle at time  $t$ ,

$$\hat{H}_{AK} = \delta(t)(\hat{\pi}_P \hat{p} + \hat{\pi}_X \hat{x}). \quad (1)$$

All other terms are neglected, in particular the ones corresponding to the free dynamics of pointers and particle. In each of these two partial interaction Hamiltonians the operator for the particle property to be measured (position  $\hat{x}$ , resp. momentum  $\hat{p}$ ) is multiplied by the conjugate operator,  $\hat{\pi}_X$ , resp.  $\hat{\pi}_P$ , of the associated pointer observable  $\hat{\mu}_X$ , resp.  $\hat{\mu}_P$ , so that for each of the ordinary von Neumann measurements the state of the pointer variable is displaced proportionally to each eigenvalue [17]. However, the combination of the two interaction terms in a single Hamiltonian implies a mutual disturbance of the two measurements.

In the combined (joint) measurement the displacement of the two commuting pointer variables  $\mu_X$  and  $\mu_P$  can be described by a true joint distribution,  $\rho(\mu_X, \mu_P)$ . It is well known that a unique position-momentum distribution for  $x$  and  $p$  (positive, bilinear in the wave function, and with the correct marginals) cannot be defined in quantum mechanics, but there are many possible quasi-distributions. The Arthurs and Kelly model provides a simple operational realization of a family of joint position-momentum quasi-distributions in terms of the commuting pointer positions as [4]

$$F(x, p) \equiv \rho(\mu_X = x, \mu_P = p). \quad (2)$$

In fact Arthurs and Kelly considered one particular set of states for the apparatus that makes  $F$  a “Husimi function” [4]; however, more general apparatus states are possible. The resulting family of quasi-distributions has been discussed by several authors [18,13].

The Arthurs and Kelly process also provides a natural way of quantizing classical functions of position and momentum. In particular, for the free motion case, Kocharński and Wódkiewicz have recently defined “operationally” a

time-of-arrival (TOA) distribution from the phase-space distribution  $\rho$  [19]. As in (2), the idea is to use the commuting variables  $\mu_X$  and  $\mu_P$ , instead of the non-commuting particle position and momentum, in the classical expression that defines the arrival time.

The theoretical treatment of time observables is an important loose end of the standard quantum mechanical formalism. Among these observables, the time of arrival has been investigated using many different approaches, as reviewed in [20] - for more recent works see [21–31]. They may be classified according to their intrinsic (ideal) or operational nature. Intrinsic models consider the particle on its own, without any external influence other than the potential in which it moves, and provide ideal quantities that do not depend on any other degree of freedom. An example is the distribution of Kijowski [32], which satisfies in a unique manner a number of properties motivated by its classical analog, or the distribution obtained within the causal theory of Bohm [33]. Operational models, instead, include extra degrees of freedom for the measuring device explicitly in the Hamiltonian [19,34,35], or implicitly by means of a non-unitary evolution law (e.g., with effective complex Hamiltonians [36,20]). Notice that the word “operational” has a rather different meaning in the usage of some authors [37], but we shall use it in the sense described. This paper investigates the properties (in particular covariance) of operational time-of-arrival distributions obtained by means of Arthurs and Kelly measurements, both of position and momentum and of the conjugate variables energy and time of arrival. In order to attain a broader perspective, we also consider and put forward other possible TOA distributions, based on different phase-space quasi-distributions, not directly derived from an Arthurs and Kelly measurement, and look into the relation between operational and ideal quantities.

In Sec. II we shall for completeness rederive the phase-space distribution  $\rho(\mu_X, \mu_P)$ , using the treatment of the Arthurs and Kelly model given by Appleby [6,7]. In Sec. III we show that the TOA distribution thus obtained, following Kochański and Wódkiewicz, is not covariant. We analyze in the following section alternative distributions, with particular emphasis on the property of covariance, and conclude that, within a broad family of phase space functions [38,39], only the Wigner function gives a covariant TOA distribution when following the recipe proposed in [19]. This covariant distribution, however, is not positive, and this defect, in turn, leads us to investigate other quantization recipes to obtain covariant and positive distributions, a task in which we succeed, using as before the simultaneous Arthurs and Kelly measurement of position and momentum. The result however is not normalizable. Yet another option in order to obtain covariant TOA distributions is to perform an Arthurs and Kelly measurement for the conjugate variables time of arrival and energy, as we show in Sec. V. This is a simple model for the important experimental need to know the energy *and* the time when the particles arrive at detectors. The results follow closely but not exactly the standard Arthurs and Kelly process for position and momentum, and we are again successful in obtaining covariant TOA distributions, now correctly normalized. In the final section of conclusions we indicate other possible extensions of our results.

## II. THE ARTHURS AND KELLY MEASUREMENT FOR POSITION AND MOMENTUM

Recently, Appleby has studied thoroughly the concepts of accuracy and disturbance in the Arthurs and Kelly model [6,7]. We shall show how to recover, following his analysis, the results for the probability  $\rho(\mu_X, \mu_P)$ , previously obtained in a concise manner by Kochański and Wódkiewicz. The interest of this detailed rederivation is that the meaning of the apparatus dependent “window” function is made explicit. The window function provides  $\rho$ , when convolved with the particle’s state, see (10) below. Our derivation also applies to a class of apparatus states more general than that in [19].

The operators for the six variables involved in the Arthurs and Kelly process satisfy the commutation relations

$$[\hat{x}, \hat{p}] = [\hat{\mu}_X, \hat{\pi}_X] = [\hat{\mu}_P, \hat{\pi}_P] = i\hbar. \quad (3)$$

Any other pair of these operators commutes (i.e.  $\hat{\mu}_P$  and  $\hat{\mu}_X$ ,  $\hat{x}$  and  $\hat{\pi}_X$ , etc...). The unitary evolution operator describing the measurement process is

$$\hat{U}_{XP} = e^{-\frac{i}{\hbar}(\hat{\pi}_P \hat{p} + \hat{\pi}_X \hat{x})}. \quad (4)$$

There is no explicit reference to the measurement instant  $t$  here.  $\hat{U}_{XP}$  may be regarded as the evolution operator connecting the states of the system before and after a sudden interaction (1). It is also possible to interpret it as the evolution operator that gives the final state for an interaction  $K(\hat{\pi}_P \hat{p} + \hat{\pi}_X \hat{x})$  acting during a time  $\Delta t = 1/K$ , with  $K$  sufficiently large so that all other terms in the Hamiltonian can be neglected during the measurement time  $\Delta t$  [1].

If the initial state of the global particle+apparatus system is given by the product state  $|\psi \otimes \psi_{ap}\rangle$ , the probability distribution for the result of the measurement takes the form

$$\rho(\mu_X, \mu_P) = \int_{-\infty}^{\infty} dx |\langle x, \mu_X, \mu_P | \hat{U}_{XP} | \psi \otimes \psi_{ap} \rangle|^2. \quad (5)$$

This is the key expression that relates the probability distribution of the pointer variables  $\mu_X$  and  $\mu_P$  to the particle's initial state. Let us emphasize that  $\rho(\mu_X, \mu_P)$  is a *true* probability distribution of its commuting variables,  $\mu_X$  and  $\mu_P$ ; however, when interpreted as a function of the *quantum mechanical* particle variables, namely as  $F(x, p) \equiv \rho(\mu_X = x, \mu_P = p)$ , it is to be understood more properly as a quasi-distribution. This comes about because the marginal distributions of  $F$  for  $x$  and  $p$  are not the quantum mechanical ones for the state of the particle, even though they are correctly normalized and positive.

To arrive at an explicit form for such a distribution and to describe the experimental errors in the measurement process, Appleby introduces the initial and final “Heisenberg picture” operators  $\hat{O}_i = \hat{O}$  and  $\hat{O}_f = \hat{U}_{XP}^\dagger \hat{O} \hat{U}_{XP}$ , where  $\hat{O}$  can be any of the six operators in (3). In terms of these operators several “errors” are defined. In particular the *retrodictive error* operators,

$$\begin{aligned} \hat{\epsilon}_{Xi} &= \hat{\mu}_{Xf} - \hat{x}_i, \\ \hat{\epsilon}_{Pi} &= \hat{\mu}_{Pf} - \hat{p}_i, \end{aligned} \quad (6)$$

provide the accuracy with which the result of the measurement reflects the state of the system *before* the measurement was carried out; and the *predictive error* operators,

$$\begin{aligned} \hat{\epsilon}_{Xf} &= \hat{\mu}_{Xf} - \hat{x}_f, \\ \hat{\epsilon}_{Pf} &= \hat{\mu}_{Pf} - \hat{p}_f, \end{aligned} \quad (7)$$

give the accuracy with which the result of the measurement reflects the state of the system *after* the measurement.

Expressions for the final operators in (6) and (7),

$$\begin{aligned} \hat{x}_f &= \hat{U}_{XP}^\dagger \hat{x} \hat{U}_{XP} = \hat{x} + \hat{\pi}_P, \\ \hat{p}_f &= \hat{U}_{XP}^\dagger \hat{p} \hat{U}_{XP} = \hat{p} - \hat{\pi}_X, \\ \hat{\mu}_{Xf} &= \hat{U}_{XP}^\dagger \hat{\mu}_X \hat{U}_{XP} = \hat{\mu}_X + \hat{x} + \frac{1}{2} \hat{\pi}_P, \\ \hat{\mu}_{Pf} &= \hat{U}_{XP}^\dagger \hat{\mu}_P \hat{U}_{XP} = \hat{\mu}_P + \hat{p} - \frac{1}{2} \hat{\pi}_X, \end{aligned} \quad (8)$$

are easily obtained using the commutation relations (3) and the following relation, valid for two arbitrary operators  $\hat{A}$  and  $\hat{B}$ ,

$$e^{\gamma \hat{A}} \hat{B} e^{-\gamma \hat{A}} = \hat{B} + \gamma [\hat{A}, \hat{B}] + \frac{\gamma^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{\gamma^3}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots \quad (9)$$

The distribution of measured values (5) can finally be written after some algebra as a convolution in phase space [7]

$$\rho(\mu_X, \mu_P) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp W_{\epsilon_i}(\mu_X - x, \mu_P - p) w(x, p), \quad (10)$$

where  $w(x, p)$  is the Wigner function of the initial state of the particle (just before the measurement takes place),

$$w(x, p) = \frac{1}{h} \int_{-\infty}^{\infty} dy e^{\frac{i}{h} p y} \left\langle x - \frac{y}{2} \right| \psi \rangle \left\langle \psi \left| x + \frac{y}{2} \right. \right\rangle, \quad (11)$$

and the apparatus dependent “window”, or “filter” function in phase space,

$$W_{\epsilon_i}(\mu_X - x, \mu_P - p) = \frac{1}{h} \int_{-\infty}^{\infty} dy e^{\frac{-i}{h} \epsilon_{Pi} y} \left\langle \epsilon_{Xi} - \frac{y}{2} \right| \hat{\rho}_{\epsilon_i} \left| \epsilon_{Xi} + \frac{y}{2} \right\rangle, \quad (12)$$

where  $\epsilon_{Pi} = \mu_P - p$ , and  $\epsilon_{Xi} = \mu_X - x$ , is the Wigner function corresponding to  $\hat{\rho}_{\epsilon_i}$ , the reduced initial apparatus state density operator for retrodictive errors,

$$\langle \epsilon_{Xi} | \hat{\rho}_{\epsilon_i} | \epsilon'_{Xi} \rangle = \int_{-\infty}^{\infty} d\epsilon_{Xf} \langle \epsilon_{Xi}, \epsilon_{Xf} | \psi_{ap} \rangle \langle \psi_{ap} | \epsilon'_{Xi}, \epsilon_{Xf} \rangle. \quad (13)$$

The trace is taken over the predictive error of position. Note that the apparatus state has been represented in the basis of the complete set of commuting operators (for the apparatus space) of retrodictive and predictive position errors,  $\hat{\epsilon}_{Xi} = \hat{\mu}_X + \hat{\pi}_P/2$ , and  $\hat{\epsilon}_{Xf} = \hat{\mu}_X - \hat{\pi}_P/2$  (that refer exclusively to the apparatus, and whose conjugate momenta are  $-\hat{\epsilon}_{Pi}$  and  $\hat{\epsilon}_{Pf}$  respectively). By changing the distribution of retrodictive errors it is possible to obtain a family of operational phase space distributions [2,40,37].

### III. TIME-OF-ARRIVAL DISTRIBUTION: INDIRECT APPROACH

By assuming a separable, pure state form for  $\hat{\varrho}_{\epsilon_i} = |\phi\rangle\langle\phi|$  in (12), and using expression (11) for  $w(x, p)$  in Eq. (10), the distribution of measured values of position and momentum can be written as

$$\rho(\mu_X, \mu_P) = \frac{1}{h} \left| \int_{-\infty}^{\infty} dx' \phi^*(\mu_X - x') \psi(x') e^{-i\mu_P x' / \hbar} \right|^2, \quad (14)$$

where the “filter” or “window” function  $\phi(\mu_X - x')$  is the retrodictive error wave function of the apparatus. (It is to be noted that while the form of (14) depends on the assumed factorized structure of the reduced retrodictive error density operator, (10) is more general and does not require such structure.) Here we may see that the quasi-distribution functions  $F(x, p) = \rho(\mu_X = x, \mu_P = p)$  are, in the language of time-frequency analysis [39], nothing but “spectrograms”, that is, the square modulus of Fourier transforms of the particle state multiplied by an apparatus-dependent window function selecting a limited spatial region. Clearly, the transform reflects the combined properties of the particle state and the window function. A natural condition to impose on  $\phi(\epsilon_{Xi})$  is that it be centered at 0, so that the average retrodictive error of position is zero, and the spatial regions where  $\psi$  and  $\rho$  have significant values are in good agreement, at least on average (since otherwise the pointer would be displaced with respect to the particle’s position). Similarly, for the momentum representation version of (14) we shall demand that the Fourier transform of the window function be also centered at zero retrodictive error of momentum. In summary, and using the language of [6], we shall assume that the measurement is “retrodictively unbiased”.

Note that in the argument of  $\phi$  the sign of  $x$  is different from the one in [19]. This can be traced back to different sign conventions for the momentum pointer and the commutation relation with its conjugate variable. Both conventions lead to equivalent results if appropriate sign changes are taken into account. With the present convention, and for unbiased measurements, the pointers are located (on average) at the particle’s average position and momentum, and thus  $\rho(\mu_X, \mu_P)$  tracks directly (of course with the unavoidable distortions inherent to the joint measurement) the particle’s position and momentum distributions. In [19], instead, the position pointer and the particle’s position have opposite signs when the filter function is centered at 0, see Eq. (21a) of [19] with  $q_0 = 0$ .

Kochański and Wódkiewicz have used the operational phase space distribution to define, *indirectly*, a TOA distribution for the free particle [19]. The basic idea is to work with the commuting operators  $\hat{\mu}_X$  and  $\hat{\mu}_P$  instead of  $\hat{x}$  and  $\hat{p}$ . A classical particle with position  $x$  and momentum  $p$  at time  $t$  takes a time  $T = -mx/p$ , measured from  $t$ , to arrive at the origin. This motivates the definition of a distribution of arrival times by means of the following average in the “observed” phase space  $\mu_X, \mu_P$ ,

$$\Pi_{KW}(T; t) \equiv \left\langle \delta \left( \frac{m\mu_X}{\mu_P} + T \right) \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\mu_X d\mu_P \delta \left( \frac{m\mu_X}{\mu_P} + T \right) \rho(\mu_X, \mu_P, t), \quad (15)$$

where  $-m\mu_X/\mu_P$  is used instead of the classical expression. (Note that  $T$  is the time interval from  $t$  to the arrival instant  $t + T$ .)

As an example, and in order to have a better grasp of the properties of this distribution, assume, as in [19], that the particle initial state is a Gaussian wave function with initial mean position  $x_0$ , initial mean momentum  $\hbar k_0$  and position “width”  $\delta$  (square root of the variance),

$$\psi(x, t) = \left( \frac{2\delta^2}{\pi} \right)^{1/4} \frac{e^{-\frac{1}{4}k_0^2\delta^2}}{\sqrt{\delta^2 + \frac{2i\hbar t}{m}}} e^{[\frac{1}{2}k_0\delta^2 + i(x-x_0)]^2 / (\delta^2 + \frac{2i\hbar t}{m})}, \quad (16)$$

and that the initial “apparatus filter-function” is given by an unbiased Gaussian with width  $\sigma$ ,

$$\phi(\mu_X - x) = \left( \frac{2}{\pi\sigma^2} \right)^{1/4} e^{-(\mu_X - x)^2 / \sigma^2}. \quad (17)$$

For the example at hand the distribution  $\rho(\mu_X, \mu_P, t)$  can be written explicitly:

$$\begin{aligned} \rho(\mu_X, \mu_P, t) &= \frac{\sigma\delta}{2\pi\hbar} \frac{1}{\sqrt{\frac{4t^2\hbar^2}{m^2} + (\delta^2 + \sigma^2)^2}} \exp \left( -\frac{1}{2\hbar^2} \frac{\delta^2\sigma^2(\delta^2 + \sigma^2)(\mu_P - \hbar k_0)^2}{\frac{4t^2\hbar^2}{m^2} + (\delta^2 + \sigma^2)^2} \right) \\ &\times \exp \left( \frac{-2[\delta^2(x_0 + \frac{\hbar k_0 t}{m} - \mu_X)^2 + \sigma^2(x_0 + \frac{\mu_P t}{m} - \mu_X)^2]}{\frac{4t^2\hbar^2}{m^2} + (\delta^2 + \sigma^2)^2} \right). \end{aligned} \quad (18)$$

Inserting this expression into (15) we obtain

$$\begin{aligned} \Pi_{KW}(T; t) = & \frac{\hbar}{2\pi m} \frac{\delta\sigma\sqrt{\frac{4t^2\hbar^2}{m^2} + (\delta^2 + \sigma^2)^2}}{\Delta(T, t)} \exp\left(\frac{-2k_0^2 [\delta^2 T_{cl}^2(t) + \sigma^2 T_{cl}^2(0) + \Delta(0, 0)]}{\frac{4t^2\hbar^2}{m^2} + (\delta^2 + \sigma^2)^2}\right) \\ & \times \left(1 + \sqrt{\pi}\xi(T, t)e^{\xi^2(T, t)}\Phi[\xi(T, t)]\right). \end{aligned} \quad (19)$$

where  $\Phi$  is the error function and the following symbols have been used:

$$\Delta(T, t) = \frac{\sigma^2(T+t)^2\hbar^2}{m^2} + \frac{\delta^2 T^2\hbar^2}{m^2} + \frac{1}{4}\delta^2\sigma^2(\delta^2 + \sigma^2) \quad (20)$$

$$T_{cl}(t) = -\frac{x_0 + \frac{k_0 t \hbar}{m}}{k_0}, \quad (21)$$

$$\xi(T, t) = \frac{2k_0[\Delta(0, 0) + \hbar(\sigma^2(T+t)T_{cl}(0) + \delta^2 T T_{cl}(t))/m]}{\sqrt{2\Delta(T, t)}\sqrt{(\delta^2 + \sigma^2)^2 + 4t^2\hbar^2/m^2}}. \quad (22)$$

This distribution is shown in Figure 1, where its lack of covariance is evident. According to the expression (19), the probability for arriving at instant  $T+t$ , when an interval  $T$  has passed after the reference time  $t$ , is not equal to the probability for arriving at the same instant  $(T+t)$ , when an interval  $T-t'$  has passed after the reference time  $t+t'$ ,

$$\Pi_{KW}(T; t) \neq \Pi_{KW}(T-t'; t+t'). \quad (23)$$

Covariance is however a basic physical requirement for any good quantum time-of-arrival distribution [41,37,21]. It simply means that the number of arrivals predicted for a particular fixed instant  $(T+t)$  in (23) should be a constant quantity independent of  $t'$ , i.e., on the reference time used in making the prediction. A good apparatus should give a stable, fixed answer, independent of the instant that we have switched it on. Lack of covariance implies that different predictions are given about the number of arrivals at the *same* instant of time depending on the reference time chosen to make the question (that corresponds here to the Arthurs and Kelly measurement). We may wonder whether a time distribution obtained from other phase-space representations of the particle state can be covariant with respect to time translations. This study is carried out in the next section.

#### IV. COVARIANCE IN TIME

The question we shall address first is whether any of the many possible representations of the quantum state, other than the spectrogram, does provide a covariant time-of-arrival distribution following the recipe shown in (15). The process of Arthurs and Kelly associates states of a particle with probability distributions on the phase space of the particle, via Eq. (2). This association, in the case of Arthurs and Kelly measurements, is done through the filter function, derived from the state of the measuring apparatus. There are however many other ways of building up such a pairing between particle state and particle phase space. (Only some of them have a simple operational interpretation in terms of a measurement model).

A very broad class of quantum quasi-probability distributions  $F$  of position and momentum was studied and defined by Cohen [38,39], the one in Eq. (14) being a particular case (see below). In Cohen's approach each of the distributions  $F$  is obtained with a different kernel  $\chi$  from the density operator  $\hat{\rho}$  of the particle, see Eq.(A1) in the appendix; and for each of these kernels a quantization rule is defined, Eq. (A4), such that the expectation values can be equally obtained by means of phase space integrals or operator traces, Eq. (A3). In particular,

$$\Pi_\delta(T; t; [\chi]) \equiv \left\langle \delta\left(\frac{mx}{p} + T\right) \right\rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp \delta\left(\frac{mx}{p} + T\right) F(x, p, t; [\chi]) = \text{Tr}[\hat{\rho}(t) \hat{\delta}_\chi(T)], \quad (24)$$

where  $\hat{\delta}_\chi(T)$  is a shorthand notation for the operator corresponding to the classical “function”  $\delta(mx/p + T)$  by means of the “ $\chi$ -quantization rule”,

$$\hat{\delta}_\chi(T) \equiv \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp d\theta d\tau \delta(T + xm/p) \chi(\theta, \tau) \exp[-i(\theta(x - \hat{x}) + \tau(p - \hat{p}))]. \quad (25)$$

Covariance in time of the arrival time distribution means that  $\Pi_\delta(T; t; [\chi])$  should be equal to

$$\Pi_\delta(T - t'; t + t'; [\chi]) = \text{Tr}[\widehat{\varrho}(t + t')\widehat{\delta}_\chi(T - t')] \quad (26)$$

$$= \text{Tr} \left[ \widehat{\varrho}(t) e^{i\widehat{H}t'/\hbar} \widehat{\delta}_\chi(T - t') e^{-i\widehat{H}t'/\hbar} \right] \quad (27)$$

for all  $\widehat{\varrho}(t)$ . Thus the covariance condition can be expressed in operator form,

$$\widehat{\delta}_\chi(T) = e^{i\widehat{H}t'/\hbar} \widehat{\delta}_\chi(T - t') e^{-i\widehat{H}t'/\hbar}. \quad (28)$$

Our next task is to find kernels  $\chi$  that fulfill (28). To this end we shall work out the momentum representation of the two sides of (28) for an arbitrary  $\chi$ . For the left hand side we find:

$$\langle p' | \widehat{\delta}_\chi(T) | p'' \rangle = \frac{1}{2\pi\hbar m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp d\tau |p| \chi \left[ \frac{p' - p''}{\hbar}, \tau \right] e^{-ip(\tau - T(p' - p'')/m\hbar)} e^{i\tau(p' + p'')/2} \quad (29)$$

$$= \frac{e^{i(p'^2 - p''^2)T/(2m\hbar)}}{2\pi\hbar m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp d\sigma |p| \chi \left[ \frac{p' - p''}{\hbar}, \sigma + \frac{T(p' - p'')}{m\hbar} \right] e^{i\sigma[p + (p' + p'')/2]}. \quad (30)$$

Use has been made of the relation

$$e^{i(\theta\widehat{q} + \tau\widehat{p})} = e^{i\hbar\theta\tau/2} e^{i\theta\widehat{q}} e^{i\tau\widehat{p}}, \quad (31)$$

and of the change of variable  $\sigma = \tau - T(p' - p'')/m\hbar$ . Operating similarly, the momentum representation of the right hand side of (28) takes the form

$$\begin{aligned} \langle p' | e^{i\widehat{H}t'/\hbar} \widehat{\delta}_\chi(T - t') e^{-i\widehat{H}t'/\hbar} | p'' \rangle &= \frac{e^{i(p'^2 - p''^2)T/(2m\hbar)}}{2\pi\hbar m} \\ &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp d\sigma |p| \chi \left[ \frac{p' - p''}{\hbar}, \sigma + \frac{(T - t')(p' - p'')}{m\hbar} \right] e^{i\sigma[p + (p' + p'')/2]}. \end{aligned} \quad (32)$$

If (32) has to be equal to (29) for all  $t'$ , the kernel function  $\chi(\theta, \tau)$  must be independent of its second argument. Since the only set of kernels worth considering are those that preserve the normalization of the state, i.e., such that  $\chi(0, 0) = 1$  [38,39], independence of  $\tau$  implies  $\chi(0, \tau) = 1$ , which is a sufficient condition to provide a correct marginal distribution for  $p$ . Therefore, independence on  $\tau$  limits the set of possible  $\chi$  kernels rather strongly, in particular the spectrogram does not generally belong to this class, since it does not generally satisfy the marginals. Its kernel has the following form, see (14),

$$\chi(\theta, \tau) = \int_{-\infty}^{\infty} dy \phi^*(y - \tau\hbar/2) \phi(y + \tau\hbar/2) e^{i\theta y}. \quad (33)$$

This may become independent of  $\tau$  in the limit of a very flat  $\phi$  function, namely, for a vanishing retrodictive error of momentum. This is not a desirable limit for our purposes: the position becomes so imprecise, that  $\Pi_{KW}(T)$  also tends to vanish in that limit, as can be seen, for example, by taking  $\sigma \rightarrow \infty$  in Eq. (19).

Restricting ourselves to the  $\chi$  functions that provide an  $F$  with the two correct marginals (this requires  $\chi(\theta, 0) = \chi(0, \tau) = 1$  [38,39], and excludes the spectrograms), the independence on  $\tau$  can only be satisfied by  $\chi = 1$ , which is the kernel that corresponds to the Wigner function and the associated Weyl quantization rule. Furthermore, in the set of the kernels which provide scale-invariance [39], characterized by being functions of the product  $\tau\theta$ ,  $\chi = 1$  is again the only possible case that presents covariance.

The covariant TOA distribution obtained with the Wigner function and (24) takes the form

$$\begin{aligned} \Pi_{\delta, Wigner}(T; t) &= \frac{1}{\hbar m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp' dp'' \left| \frac{p' + p''}{2} \right| \langle p' | \widehat{\varrho}(t + T) | p'' \rangle \\ &= \Pi_{\delta, Wigner}(T - t'; t + t'). \end{aligned} \quad (34)$$

Let us compare this result with the flux at the origin,  $J = \int_{-\infty}^{\infty} dp w(0, p, t) \frac{p}{m}$ . By substituting (11), and performing several integrals, this takes the form

$$J = \frac{1}{\hbar m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dp' dp'' \left( \frac{p' + p''}{2} \right) \langle p' | \widehat{\varrho}(t) | p'' \rangle. \quad (35)$$

The only difference with (34) is the presence or absence of the absolute value in the half-sum, the two results being equal for states without negative momentum components. However, a quantum mechanical state composed by positive momenta is compatible with a negative value of  $J$  (backflow) at certain times and positions [42,43,27]. Therefore, the “time-of-arrival quasi-distribution” obtained by means of Wigner’s function satisfies the covariance under time translations, but fails to be a positive distribution, which is also an important requirement for a good time-of-arrival distribution.

### A. Kijowski's distribution

It is also of interest to investigate whether there is a kernel function  $\chi$  such that the corresponding distribution  $\Pi(T)$  obtained via (24) is equal to the covariant distribution of Kijowski,

$$\Pi_K(T; t) = \text{Tr} \left[ \hat{\varrho}(t) \left( \sum_{\alpha} |T, \alpha\rangle \langle T, \alpha| \right) \right]. \quad (36)$$

Here  $|T, \alpha\rangle$  are the (generalized) eigenfunctions of the Aharonov-Bohm time-of-arrival operator (see below),

$$\langle p|T, \alpha\rangle = \left( \frac{|p|}{m\hbar} \right)^{1/2} e^{ip^2 T/2m\hbar} \Theta(\alpha p), \quad (37)$$

and  $\alpha = \pm$  is the degeneracy index associated with positive or negative momentum.

We look for a kernel  $\chi$  such that  $\hat{\delta}_{\chi}(T) = \sum_{\alpha} |T, \alpha\rangle \langle T, \alpha|$ . Working again in momentum representation,

$$\langle p' | \hat{\delta}_{\chi}(T) | p'' \rangle = \Theta(p' p'') \frac{(p' p'')^{1/2}}{m\hbar} e^{i(p'^2 - p''^2)T/(2m\hbar)} e^{i(p' - p'')X/\hbar}. \quad (38)$$

Comparing with (29) we see that  $\chi$  need not depend on its second argument, as should have been expected since  $\Pi_K$  is covariant. The integrals in (29) can be then carried out, and the requirement (38) becomes

$$\chi \left( \frac{p' - p''}{\hbar} \right) = \Theta(p' p'') (p' p'')^{1/2} \left| \frac{2}{p' + p''} \right|. \quad (39)$$

Changing variables to  $\nu = p' - p''$  and  $\eta = (p' + p'')/2$ , it is clear that no function of  $\nu$  can satisfy this equation, since the right hand side also depends on  $\eta$ . It follows that Kijowski's distribution cannot be obtained by such an extension of the procedure of Kočański and Wódkiewicz.

### B. Alternative quantization of time of arrival

Up to now we have used position-momentum representations of the particle state to define time distributions by means of (24), i.e., by an extension of the procedure proposed by Kočański and Wódkiewicz. Let us now show that this does not exhaust all possible quantizations of the classical distribution, and in fact that it is possible to do much better with respect to covariance and positivity. Consider the following quantities,

$$\Pi_{\tilde{J}}(T; t; [\chi]) \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \left| \frac{p}{m} \right| \delta(x) F(x, p; T + t; [\chi]), \quad (40)$$

where the subscript  $\tilde{J}$  indicates that in the classical version of this expression, where  $F$  would be a true phase space joint distribution,  $\Pi_{\tilde{J}}$  represents the positive minus the negative fluxes (the positive flux is defined by  $\int_0^{\infty} dp F(0, p)p/m$ , and the negative flux by  $\int_{-\infty}^0 dp F(0, p)p/m$ ). This is apparently quite different from (24) but, in fact, in the context of classical mechanics of the free particle, the two expressions are equivalent, as can be seen by using Liouville's theorem and the trajectory equation for free motion. However, their quantizations are not the same in general. Note in particular that in (40) the time dependence has been put entirely in the state and not in the observable. An important consequence is that the functions  $\Pi_{\tilde{J}}(T; t; [\chi])$  are time covariant *for all*  $\chi$ . A case where (24) and (40) are equal is  $\chi = 1$ , because the propagator of the Wigner function for free motion is just the classical propagator. In general, (40) can be understood as a "covariantization" of (24): they coincide for  $T = 0$ , and (40) is covariant by construction, which means that

$$\Pi_{\tilde{J}}(T; t; [\chi]) = \Pi_{\tilde{J}}(0; t + T; [\chi]) \quad (41)$$

for all  $t$  and  $T$ , and function  $\chi$ .

The difference between (40) and (24) is actually more profound: whereas in (24) we are performing *quantization of a family* of classical functions, parameterised by  $T$ , namely,  $\delta(mx/p + T)$ , (40) pertains to a *family of quantizations* for the *same* classical function,  $\left| \frac{p}{m} \right| \delta(x)$ , each of the quantizations being parameterised by  $T$ . Since the value of Kijowski's distribution for a given  $T$  is actually the trace with the state of a suitable operator, it seems unlikely that

it can be reproduced within a scheme so radically different in conceptual terms as a family of quantizations, and, in fact, using a procedure similar to the one in the previous subsection, one can prove that Kijowski's distribution cannot be obtained from the quantization of  $|\frac{p}{m}|\delta(q)$  with any  $\chi$  within the class being considered [44].

Even so, positive TOA distributions can be obtained from expression (40) for suitable quantization functions  $\chi$ . Indeed, a sufficient condition is that  $F[\chi]$  be positive. In particular, the spectrogram is positive and provides a covariant TOA distribution via (40). For the example worked out before, see (18),

$$\begin{aligned} \Pi_j(T; t) = & \frac{\hbar}{2\pi m} \frac{\delta\sigma \sqrt{\frac{4(t+T)^2\hbar^2}{m^2} + (\delta^2 + \sigma^2)^2}}{\Delta(0, t+T)} \exp\left(\frac{-2k_0^2[\delta^2 T_{cl}^2(t+T) + \sigma^2 T_{cl}^2(0) + \Delta(0, 0)]}{\frac{4(t+T)^2\hbar^2}{m^2} + (\delta^2 + \sigma^2)^2}\right) \\ & \times \left(1 + \sqrt{\pi}\xi(0; T+t)e^{\xi^2(0; T+t)}\Phi[\xi(0; T+t)]\right). \end{aligned} \quad (42)$$

This distribution is represented in Figure 1. The problem is that, due to the asymptotic dependence  $\sim 1/|T|$  for large  $|T|$ , it cannot be normalized. In fact whereas by construction (24) is normalized (provided the phase space density is normalized), (40) is not automatically normalized for an arbitrary  $\chi$ , and can actually be non-normalizable as in this example. Positive, covariant, and normalized operational TOA distributions will be obtained in the following section.

## V. ARTHURS AND KELLY MODEL FOR TIME OF ARRIVAL AND ENERGY

In the previous section TOA distributions have been defined indirectly from position-momentum quasi-distributions. Similar energy distributions can also be obtained indirectly, as pointed out in [19]. However, an Arthurs and Kelly type of measurement can be used to find operationally TOA and energy distributions in a *direct* way: instead of the conjugate variables position and momentum, we will describe the measurement process in terms of the conjugate variables energy  $E$  and time of arrival  $T$ . (In this section we shall use the somewhat unusual notation  $\hat{E}$  for the free motion Hamiltonian of the particle  $\hat{E} = \hat{H} = \hat{p}^2/(2m)$  to keep the notation in parallel with the  $\{x, p\}$  measurement model.) In our  $\{E, T\}$  model the operator  $\hat{T}$  is taken as the time operator introduced by Aharonov and Bohm [45],

$$\hat{T} = -\frac{m}{2} \left( \hat{x} \frac{1}{\hat{p}} + \frac{1}{\hat{p}} \hat{x} \right), \quad (43)$$

by symmetrizing the classical expression,  $-mx/p$ , for the time of arrival at  $X = 0$ , computed from  $t$ , of a particle that at time  $t$  has position  $x$  and momentum  $p$ . This operator is not self-adjoint but maximally symmetric, see a detailed discussion in [28]. Its (generalized) eigenfunctions in momentum representation are given by (37). The  $\{E, T\}$  Arthurs and Kelly measurement process is based on pointer and particle operators,  $\hat{\mu}_T, \hat{\mu}_E, \hat{\pi}_T, \hat{\pi}_E, \hat{E}$ , and  $\hat{T}$ , parallel to the set used for  $x$  and  $p$ , and related by similar commutation relations,

$$[\hat{E}, \hat{T}] = [\hat{\mu}_E, \hat{\pi}_E] = [\hat{\mu}_T, \hat{\pi}_T] = i\hbar, \quad (44)$$

with all other commutators being zero.

The energy operator is bounded from below. As explained later it is also necessary to require that the operator  $\hat{\pi}_T$  be bounded from below. Another difference with the  $\{x, p\}$  case is the degeneracy, associated with positive and negative momenta, of the spectra of  $\hat{E}$  and  $\hat{T}$ . We shall keep the pointer variables non-degenerate for simplicity although a more detailed model including pointers sensitive to the degeneracy index is also possible.

The evolution operator describing the measurement process is now given by

$$\hat{U}_{ET} = e^{-\frac{i}{\hbar}(\pi_T \hat{T} + \pi_E \hat{E})}. \quad (45)$$

Our purpose is yet again to obtain the probability distribution for the result of the measurement,

$$\rho(\mu_E, \mu_T) = \sum_{\alpha} \int_0^{\infty} dE |\langle E, \alpha, \mu_E, \mu_T | \hat{U}_{ET} | \psi \otimes \psi_{ap} \rangle|^2. \quad (46)$$

In Sec. II, in the  $\{x, p\}$  phase space, the error operators and all their relations were obtained using the definitions of final and initial operators, the commutation relations (3), and expression (9). In the  $\{E, T\}$  phase space similar relations are also valid, and (6) to (8) hold by substituting  $E$  for  $x$  and  $T$  for  $p$ .



In order to construct a meaningful Arthurs and Kelly model for TOA and energy, it is useful to analyze first the simple, but not trivial, von Neumann measurement model of  $\hat{T}$ , corresponding to the operator  $\exp -i\hat{\pi}_T\hat{T}/\hbar$ . Note that the basis  $\{|T, \alpha\rangle\}$  is complete but non-orthogonal, because  $\hat{T}$  is not self-adjoint. However, the intermediate computations are mostly carried out in the  $\{|E, \alpha\rangle\}$  basis, which is indeed complete and orthogonal, since it corresponds to the spectral decomposition of a self-adjoint operator. Using the overlap

$$\langle E, \alpha | T, \alpha' \rangle = \frac{1}{h^{1/2}} e^{iET/\hbar} \delta_{\alpha\alpha'}, \quad (47)$$

we find

$$\langle E, \alpha, \pi_T | e^{-\frac{i}{\hbar}\hat{\pi}_T\hat{T}} | E', \alpha', \pi'_T \rangle = \delta(E - \pi_T - E') \delta_{\alpha\alpha'} \delta(\pi_T - \pi'_T). \quad (48)$$

When integrating over  $E$ , which is a positive variable, the energy delta function cannot be satisfied unless  $E' + \pi_T$  is positive. This means that the exponential displaces the energy of the energy eigenstate as long as  $E' + \pi_T$  remains positive, but annihilates the state otherwise,

$$e^{-\frac{i}{\hbar}\hat{\pi}_T\hat{T}} | E', \alpha' \rangle = \Theta(E' + \pi_T) | E' + \pi_T, \alpha' \rangle. \quad (49)$$

It would be more rigorous to deal with wave packets in the energy representation, given that  $\hat{T}$  is not self-adjoint. However, the results we will be using are actually unchanged, so there is no need to insist on this fact. In order to guarantee that the exponential operator is isometric (i.e. preserves norm), we shall restrict the spectrum of  $\pi_T$  to the positive half line.  $\hat{\pi}_T$  and  $\hat{\mu}_T$  are thus a conjugate pair of operators similar to  $\hat{E}$  and  $\hat{T}$  except for the degeneracy. In particular, this means that  $\hat{\mu}_T$  is not self-adjoint, because of Pauli's theorem [46,28]. Using similar arguments for each of the exponentials in

$$\hat{U}_{ET} = e^{-\frac{i}{\hbar}(\hat{\pi}_T\hat{T} + \hat{\pi}_E\hat{E})} = e^{-\frac{i}{2\hbar}\hat{\pi}_T\hat{\pi}_E} e^{-\frac{i}{\hbar}\hat{\pi}_T\hat{T}} e^{\frac{i}{\hbar}\hat{\pi}_E\hat{E}}, \quad (50)$$

it is easily proved that  $\hat{U}_{ET}$  is isometric provided that  $\hat{\pi}_T$  is positive, and irrespective of the (real) spectrum chosen for  $\hat{\mu}_E$ . Indeed, there would be two different possible models, depending on whether  $0 \leq \mu_E < \infty$  or  $-\infty < \mu_E < \infty$  is chosen. The bounded case has the advantage of strictly positive values, mimicking the restriction on values of  $E$ , but the disadvantage that initial apparatus states near the origin cannot be Gaussian nor symmetrically peaked around  $\mu_E = 0$ . For the unbounded case, the initial state may be located symmetrically around the origin, however, the price of there being negative values has to be paid. We shall work out the second possibility hereafter, but, with little changes, the analysis of the alternative apparatus may be also carried out. In particular, the important property of time covariance is valid for both cases. Let us first see how  $\hat{U}_{ET}$  acts on a convenient basis for the composite system:

$$\hat{U}_{ET} | E, \alpha, \mu_E, \pi_T \rangle = \left| E + \pi_T, \alpha, \mu_E + E + \frac{\pi_T}{2}, \pi_T \right\rangle. \quad (51)$$

By inserting a resolution of the identity in this basis one finds

$$\langle E, \alpha, \mu_E, \pi_T | \hat{U}_{ET} | \psi \otimes \psi_{ap} \rangle = \left\langle E - \pi_T, \alpha, \mu_E - E + \frac{\pi_T}{2}, \pi_T \right| \psi \otimes \psi_{ap} \rangle \Theta(E - \pi_T), \quad (52)$$

whose Fourier transform (to change from  $\pi_T$  to the conjugate variable  $\mu_E$ ) is, using now the basis of retrodictive and predictive energy errors for the apparatus state, and  $E' = E - \pi_T$ ,

$$\langle E, \alpha, \mu_E, \mu_T | \hat{U}_{ET} | \psi \otimes \psi_{ap} \rangle = \frac{1}{h^{1/2}} \int_0^E dE' e^{\frac{i}{\hbar}[\mu_T(E-E')]} \langle \mu_E - E', \mu_E - E | \psi_{ap} \rangle \langle E', \alpha | \psi \rangle. \quad (53)$$

The upper limit reflects the fact that the measurement always shifts the particle's energy "upwards". Any resulting energy component  $\langle E |$  can only have contributions from the lower energy states. From here the operational joint time-of-arrival and energy distribution takes the form

$$\begin{aligned} \rho[\mu_E, \mu_T; \psi(t=0)] &= \frac{1}{h} \sum_{\alpha} \int_0^{\infty} dE \int_0^E dE' \int_0^E dE'' e^{\frac{i}{\hbar}\mu_T(E''-E')} \\ &\times \langle \mu_E - E', \mu_E - E | \psi_{ap} \rangle \langle \psi_{ap} | \mu_E - E'', \mu_E - E \rangle \langle E', \alpha | \psi(0) \rangle \langle \psi(0) | E'', \alpha \rangle. \end{aligned} \quad (54)$$

Let us now examine some properties of this distribution: It is time covariant as can be easily seen from (54):

$$\begin{aligned} \rho[\mu_E, \mu_T - t; \psi(t)] &= \sum_{\alpha} \frac{1}{h} \int_0^{\infty} dE \int_0^E dE' \int_0^E dE'' e^{\frac{i}{\hbar}(\mu_T - t)(E'' - E')} \\ &\times \langle \mu_E - E', \mu_E - E | \psi_{ap} \rangle \langle \psi_{ap} | \mu_E - E'', \mu_E - E \rangle \langle E', \alpha | e^{-iE't/\hbar} | \psi(0) \rangle \langle \psi(0) | e^{iE''t/\hbar} | E'', \alpha \rangle \\ &= \rho[\mu_E, \mu_T; \psi(0)], \end{aligned} \quad (55)$$

because of the cancellation of the  $t$ -dependent exponentials. It is also possible to write it in a form similar, but not identical, to Eq. (10). To this end we shall define the following apparatus dependent object in the basis of retrodictive error of energy,

$$\langle a | \hat{\varrho}_{ap}(c) | b \rangle \equiv \int_0^{\infty} dE \langle a, c - E | \psi_{ap} \rangle \langle \psi_{ap} | b, c - E \rangle \Theta(E - c + a) \Theta(E - c + b). \quad (56)$$

Changing variables to the half-sum and difference,

$$\begin{aligned} s &= \frac{E' + E''}{2} \\ y &= E'' - E', \end{aligned} \quad (57)$$

we can write (54) as

$$\begin{aligned} \rho(\mu_E, \mu_T) &= \sum_{\alpha} \frac{1}{h} \int_0^{\infty} ds \int_{-\infty}^{\infty} dy e^{i\mu_T y/\hbar} \langle \mu_E - s + y/2 | \hat{\varrho}_{ap}(\mu_E) | \mu_E - s - y/2 \rangle \\ &\times \langle s - y/2, \alpha | \psi \rangle \langle \psi | s + y/2, \alpha \rangle \Theta(s - y/2) \Theta(s + y/2). \end{aligned} \quad (58)$$

Alternatively,

$$\rho(\mu_E, \mu_T) = \sum_{\alpha} \int_0^{\infty} dE \int_{-\infty}^{\infty} dT W_{\mu_E}(\mu_E - E, \mu_T - T) w_{\alpha, \alpha}(E, T), \quad (59)$$

where

$$w_{\alpha, \alpha'}(E, T) = \frac{1}{h} \int_{-\infty}^{\infty} dy e^{iTy/\hbar} \langle E - y/2, \alpha | \psi \rangle \langle \psi | E + y/2, \alpha' \rangle \Theta(E - y/2) \Theta(E + y/2), \quad (60)$$

and

$$W_{\mu_E}(\mu_E - E, \mu_T - T) = \frac{1}{h} \int_{-\infty}^{\infty} dy e^{-i(\mu_T - T)y/\hbar} \langle \mu_E - E - y/2 | \hat{\varrho}_{ap}(\mu_E) | \mu_E - E + y/2 \rangle, \quad (61)$$

as can be checked by substitution of (60) and (61) into (59), and renaming  $s = E$  in (58).  $W_{\mu_E}$  is an apparatus dependent window function for the energy-TOA distribution of the particle. Even though (61) and (56) provide an explicit expression, its interpretation is not as simple as the corresponding phase space window function for the  $\{x, p\}$  case of Sec. II. Because of the lower energy bound of the energy,  $\hat{\varrho}_{ap}(\mu_E)$  is not, in general, the reduced density operator of the apparatus for the retrodictive errors. The filtered function,  $w_{\alpha, \alpha}(E, T)$ , has a simpler content as an  $\alpha, \alpha$  diagonal component of the energy-TOA Wigner matrix for the particle, Eq. (60). Remarkably, no interference term with  $\alpha \neq \alpha'$  contributes to (59), a feature shared with Kijowski's time-of-arrival distribution. As a matter of fact, by tracing over  $\alpha$  and integrating over  $E$ , the marginal of the particle's Wigner matrix is nothing but Kijowski's distribution,

$$\begin{aligned} \sum_{\alpha} \int_0^{\infty} dE w_{\alpha, \alpha}(E, T) &= \frac{1}{h} \sum_{\alpha} \int_0^{\infty} dE' \int_0^{\infty} dE'' e^{iT(E'' - E')/\hbar} \langle E', \alpha | \psi \rangle \langle \psi | E'', \alpha \rangle \\ &= \sum_{\alpha} \langle T, \alpha | \psi \rangle \langle \psi | T, \alpha \rangle = \Pi_K(T), \end{aligned} \quad (62)$$

as is readily seen by undoing the change of variables displayed in (57) and using Eq. (47). However, whereas in the von Neumann measurement the resulting distribution  $P(\mu_T)$  is a smoothed version of Kijowski's distribution,

see Appendix B, the TOA marginal of the Arthurs and Kelly distribution obtained by integrating (59) over  $\mu_E$ , not only smooths but also distorts Kijowski's distribution. This is due to the double dependence on  $\mu_E$  of the apparatus dependent window function  $W_{\mu_E}(\mu_E - E, \mu_T - T)$ : aside from  $w_{\alpha,\alpha}(E, T)$  the integral over  $\mu_E$  leaves an extra  $E$ -dependent function. This is one further peculiarity of the  $\{E, T\}$  Arthurs and Kelly model with respect to the  $\{x, p\}$  model of Sec. II. In the later, there is no double dependence on  $\mu_X$  in the apparatus window Wigner function, and the marginal is simply a smoothed version of the quantum mechanical distribution of the particle's position, without additional distortion.

## VI. DISCUSSION

In this work we have explored different ways to obtain time-of-arrival (TOA) distributions “operationally”, i.e., by means of models that include the particle and additional degrees of freedom acting as meters or sensors of the particle. An indirect route is to *define* the TOA distribution from a distribution of position and momentum that is determined operationally. The Arthurs and Kelly measurement model is possibly the simplest model where such distributions may be obtained. We have shown however that in order to obtain a time covariant result the step from the phase space distribution to the TOA distribution is delicate, even crucial. Furthermore, when we consider generic distributions of position and momentum the essential property of time covariance proves to be rather elusive. In particular, we have proved that for the choice of classical phase space function  $\delta(T + mq/p)$  (that is, the choice of Kochański and Wódkiewicz, [19]) there is no distribution of position and momentum within a wide class for which the associated TOA distribution is covariant in time. Moreover, we have given yet another instance of the well known fact that two classically equivalent expressions may give different quantum distributions, which, in the case at hand, are covariant or not. Thus, we have proposed a different classical phase space function,  $\delta(q)|p/m|$ , for which covariance is always granted for a wide class of phase space families of distributions parameterized by  $T$ , compare (24) and (40). Not only are the quantum distributions numerically different from the classical ones, conceptually they are also worlds apart: whereas the quantization of Kochański and Wódkiewicz should be more properly understood as the quantization, within the same quantization scheme, of a family of classical functions, our proposal is rather a family of quantization schemes of the same classical function.

The question now to be posed is whether any of these different indirect methods of obtaining TOA distributions can be selected as being “better” in some suitable sense. From the theoretical point of view, covariance and positivity seem to be minimal requirements, therefore selecting positive flux ( $\delta(q)|p/m|$ ) over trajectory identification ( $\delta(T + mq/p)$ ).

Nonetheless, in the case of the free particle, not even the quantization of positive flux provides us with the “best” TOA distribution, in that it may be not normalizable, as we have shown with an example, and that for any quantization of positive flux the variance of the distribution will be larger than for Kijowski's distribution, which we have shown does not lie within a wide class of quantizations of  $\delta(q)|p/m|$ . We are in this manner led to propose a different operational approach, in which time of arrival and energy are directly the “measured” variables. In this case time covariance is automatically satisfied. Not only that: Kijowski's TOA distribution can now be understood as a marginal distribution deriving from the Wigner function, suitably generalized for the pair of variables energy and time of arrival. Operationally, a simple TOA von Neumann measurement provides a smoothed Kijowski's distribution whereas the marginal of the time-energy distribution that results from the Arthurs and Kelly process is distorted beyond a simple smoothing. In both cases and for a generic apparatus state we see that the variance of the TOA distributions hence derived is bigger than for Kijowski's, as was only to be expected from the axiomatic derivation of Kijowski's distribution, which selects  $\Pi_K$  as the one with smallest variance [32].

Additional to our central objective, namely, the study of operational definitions of TOA distributions and their relation to ideal measurements of time, we have also obtained a general result concerning operational definitions of measurements of pairs of conjugate variables when one of them has a bounded spectrum from below. For instance,  $x$  and  $p$  in the half line. The naïve construction of Wigner's function for such a case is actually valid, even though  $\hat{p}$  is not self-adjoint. The reason for the validity of the naïve expression is, however, far from simple. Since  $\hat{p}$  is not self-adjoint and admits no self-adjoint extension on the half-line (technically, it is a maximally symmetric operator with  $(1, 0)$  defect indices), there is no spectral decomposition available for it. Even so, we do have the next best thing, namely, a positive operator valued measure (POVM) [37,28], and this allows the construction of the basis of operators required for the definition of Wigner's function. Moreover, the probability density obtained as the marginal distribution for the naïvely defined Wigner function can be used to reconstruct the whole POVM. To the best of our knowledge, the applicability of the Wigner function formalism and its generalizations has not been discussed heretofore in the literature for this case.

On a more general note, we would like to point out that even though the world of “operational models” is in principle between the ideal theories (depending only on the particle's state) and actual experiments, the connections are not

always explicit or obvious. We have in particular emphasized that the phase space probability obtained operationally by means of the position-momentum Arthurs and Kelly process may be regarded as one particular (ideal) joint quasi-distribution. Many others exist, in particular the ones in the family defined by Cohen, but their links with a particular operational procedure are not generally as direct. The fact that Kijowski's ideal TOA distribution, the closest object to the classical TOA distribution considering the common properties satisfied, is not obtained with any of these phase space quasi-distributions indicates that there is a limit to the flexibility of the class of quantization rules included in Cohen's formalism, a warning flag worth considering when dealing with other quantization problems.

The other connection, between operational results and actual experiments is sometimes taken for granted, but in fact it is far from being easily realized. The conjugate variables measured by "Arthurs and Kelly experiments" in quantum optics are the two quadrature components of the electric field strength of a single-mode radiation field, not the position and momentum of a particle. Nevertheless we expect a similar experiment with particles to be feasible soon (see [47]). In future work we also intend to relate the ideal TOA distributions with operational models that describe a continuous (rather than instantaneous) measurement process, thus being closer to an actual time-of-flight experiment as emphasized by Aharonov et al. [34].

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## APPENDIX A: BASIC COHEN'S PHASE SPACE FORMALISM

Given an operator  $\hat{G}$  and a density operator  $\hat{\rho}$  their phase space representatives

$$F(x, p) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du \left\langle u + \frac{\hbar\tau}{2} \left| \hat{\rho} \right| u - \frac{\hbar\tau}{2} \right\rangle e^{-i[\theta(x-u)+\tau p]} \chi(\theta, \tau) \quad (A1)$$

$$g(q, p) = \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du \left\langle u - \frac{\hbar\tau}{2} \left| \hat{G} \right| u + \frac{\hbar\tau}{2} \right\rangle \frac{e^{i[\theta(q-u)+\tau p]}}{\chi(\theta, \tau)} \quad (A2)$$

are chosen such that

$$\langle \hat{G}(\hat{x}, \hat{p}) \rangle = \text{tr}(\hat{\rho} \hat{G}) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp F(x, p) g(x, p). \quad (A3)$$

Conversely, given two phase space representatives for the state and dynamical variable, their operators are obtained by

$$\hat{\rho} = \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\tau F(x, p) \chi(\theta, \tau)^{-1} e^{i[\theta(x-\hat{x})+\tau(p-\hat{p})]} \quad (A4)$$

$$\hat{G}(\hat{x}, \hat{p}) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} d\theta \int_{-\infty}^{\infty} d\tau g(x, p) e^{-i(\theta x + \tau p)} \chi(\theta, \tau) e^{i(\theta \hat{x} + \tau \hat{p})}. \quad (A5)$$

## APPENDIX B: VON NEUMANN MEASUREMENT OF $\hat{T}$

Using (49) the amplitude that results from the von Neumann measurement of the time operator  $\hat{T}$  is given, for a semibounded  $\hat{\pi}_T$ , by

$$\langle T, \alpha, \mu_T | e^{-i\hat{\pi}_T \hat{T}/\hbar} \psi(t) \otimes \psi_{ap} \rangle = \langle \mu_T - T | \psi_{ap} \rangle \langle T, \alpha | \psi(t) \rangle. \quad (B1)$$

Since there is only one pointer the wave function of the apparatus,  $\psi_{ap}$ , depends only on one variable here. The probability to find  $\mu_T$  in a measurement carried out at  $t$  is a smoothed version of Kijowski's distribution, see (36),

$$P(\mu_T; t) = \int_{-\infty}^{\infty} dT |\langle \mu_T - T | \psi_{ap} \rangle|^2 \Pi_K(T; t), \quad (B2)$$

which is covariant [ $P(\mu_T - t'; t + t') = P(\mu_T; t)$ ] because of the covariance of  $\Pi_K$ .

- [1] E. Arthurs and J. L. Kelly, Bell Syst. Tech. J. **44**, 1153 (1965).
- [2] Leonhardt U. *Measuring the Quantum State of Light*, (Cambridge University Press, Cambridge, 1997).
- [3] S. L. Braunstein, C. M. Caves and G. J. Milburn, Phys. Rev A **43**, 1153 (1991).
- [4] S. Stenholm, Ann. Phys. **218**, 233 (1992).
- [5] P. Kočański and K. Wódkiewicz, Rep Math. Phys. **40**, 245 (1997).
- [6] D. M. Appleby, Int. J. Theor. Phys. **37**, 1491 (1998).
- [7] D. M. Appleby, J. Phys. A **31**, 6419 (1998).
- [8] H. Martens and W. M. de Muynck, Found. Phys. **20**, 357 (1990).
- [9] W. M. de Muynck, quant-ph 9901010.
- [10] P. Busch, Int. J. Theor. Phys. **24**, 63 (1982).
- [11] Y. Aharonov, D. Z. Albert, and C. K. Au, Phys. Rev. Lett. **47**, 1029 (1981).
- [12] R. F. O'Connell and A. K. Rajagopal, Phys. Rev. Lett. **48**, 525 (1982).
- [13] K. Wódkiewicz, Phys. Rev. Lett. **52**, 1064 (1984).
- [14] J. J. Halliwell, Physical Review D **46**, 1610 (1992).
- [15] P. Törmä, S. Stenholm and I. Jex, Phys. Rev. A **52**, 4812 (1995).
- [16] G. S. Agarwal and S. Chaturvedi, Phys. Rev. A **49**, R665 (1994).
- [17] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).
- [18] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976).
- [19] P. Kočański and K. Wódkiewicz, Phys. Rev. A **60**, 2689 (1999).
- [20] J. G. Muga, R. Sala, and J. P. Palao, Superlattices and Microstructures **23**, 833 (1998). See also J. G. Muga and C. R. Leavens, *Arrival time in quantum mechanics*, in preparation.
- [21] J. G. Muga, C. R. Leavens and J. P. Palao, Phys. Rev. **58**, 4336 (1998).
- [22] J. Finkelstein, Phys. Rev. A **59**, 3218 (1999).
- [23] M. Toller, Phys. Rev. A **59**, 960 (1999).
- [24] J. Kijowski, Phys. Rev. A **59**, 897 (1999).
- [25] J. Oppenheim, B. Reznik, and W. G. Unruh, Phys. Rev. A **59**, 1804 (1999).
- [26] V. Delgado, Phys. Rev. A **59**, 1010 (1999).
- [27] J. G. Muga, J. P. Palao, C. R. Leavens, Phys. Lett. A **253**, 21 (1999).
- [28] I. Egusquiza and J. G. Muga, Phys. Rev. A (1999), accepted, quant-ph 9904055.
- [29] A. D. Baute, R. Sala Mayato, J. P. Palao, J. G. Muga and I. L. Egusquiza, Phys. Rev. A **61** (2000), accepted, quant-ph 9904055.
- [30] J. León, J. Julve, P. Pitanga and F. J. Urriés, quant-ph 9903060.
- [31] E. Galapón, quant-ph 9908033.
- [32] J. Kijowski, Rep. Math. Phys. **6**, 362 (1974).
- [33] C. R. Leavens, Phys. Rev. A **58**, 840 (1998).
- [34] Y. Aharonov, J. Oppenheim, S. Popescu, B. Reznik, and W. G. Unruh, Phys. Rev. A **57**, 4130 (1998).
- [35] J. Halliwell, quant-ph/9805057, to appear in Prog. Th. Phys. 102 (1999).
- [36] J. G. Muga, S. Brouard and D. Macías, Annals of Physics **240**, 351 (1995).
- [37] P. Busch, M. Grabowski and P. J. Lathi, *Operational Quantum Mechanics* (Springer-Verlag, Berlin, 1995).
- [38] L. Cohen, Jour. Math. Phys. **7**, 781 (1966).
- [39] L. Cohen, *Time-Frequency Analysis* (Prentice Hall, New York, 1995).
- [40] E. B. Davies, *Quantum Theory of Open System* (Academic Press, New York, 1976).
- [41] R. Werner, J. Math. Phys. **27**, 793 (1986).
- [42] G. R. Allcock, Ann. Phys. (NY) **53**, 253 (1969); G. R. Allcock, Ann. Phys. (NY) **53**, 286 (1969); G. R. Allcock, Ann. Phys. (NY) **53**, 311 (1969).
- [43] A. J. Bracken and G. F. Mello, J. Phys. **A27**, 2197 (1994).
- [44] We only consider  $\chi$  kernels which are state-independent so that the associated quantization rules are also state independent, and the phase space representatives of the quantum state depend bilinearly on the wave function. Other possibilities have been considered by Cohen and coworkers, see L. Cohen and Y. Zaporovanny, J. Math. Phys. **21**, 794 (1980).
- [45] Y. Aharonov and D. Bohm, Phys. Rev. A **122**, 1649 (1961).
- [46] W. Pauli, Die allgemeinen prinzipien der Wellenmechanik. Handbuch der Physik, ed. S. Flüge, vol. V/1, Springer-Verlag (1958); English version: General Principles of Quantum Mechanics, Springer-Verlag, Berlin (1980) (p. 63).
- [47] W. L. Power, S. M. Tan, and M. Wilkens, J. Mod. Opt. **44** (1997) 2591.

FIGURE CAPTIONS:

**Figure 1:**  $\Pi_{KW}(T - t; t)$  versus  $T$ :  $t = -0.2$  (long dashed line);  $t = -0.1$  (short dashed line).  $\Pi_{\tilde{J}}(T - t; t)$  versus  $T$  for any  $t$ , (solid line). In all cases  $x_0 = -2.5$ ,  $k_0 = 10$ ,  $\sigma = \delta = 0.1$ , and  $\hbar = m = 1$  (atomic units). The bumps on the left/right are essentially due to contributions from negative/positive momenta.